Activity Report 2019

Project-Team TROPICAL

Tropical methods: structures, algorithms and interactions

IN COLLABORATION WITH: Centre de Mathématiques Appliquées (CMAP)

RESEARCH CENTER
Saclay - Île-de-France

THEME
Optimization and control of dynamic systems

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Project-Team TROPICAL

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Keywords:

Computer Science and Digital Science:

A1.2.4. - QoS, performance evaluation
A2.3.3. - Real-time systems
A2.4. - Formal method for verification, reliability, certification
A6.2.5. - Numerical Linear Algebra
A6.2.6. - Optimization
A6.4.6. - Optimal control
A7.2.4. - Mechanized Formalization of Mathematics
A8.1. - Discrete mathematics, combinatorics
A8.2.1. - Operations research
A8.2.3. - Calculus of variations
A8.3. - Geometry, Topology
A8.9. - Performance evaluation
A8.11. - Game Theory
A9.6. - Decision support

Other Research Topics and Application Domains:

B4.3. - Renewable energy production
B4.4. - Energy delivery
B4.4.1. - Smart grids
B6.6. - Embedded systems
B8.4. - Security and personal assistance
B8.4.1. - Crisis management

1. Team, Visitors, External Collaborators

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2. Overall Objectives

2.1. Introduction

The project develops tropical methods motivated by applications arising in decision theory (deterministic and stochastic optimal control, game theory, optimization and operations research), in the analysis or control of classes of dynamical systems (including timed discrete event systems and positive systems), in the verification of programs and systems, and in the development of numerical algorithms. Tropical algebra tools are used in interaction with various methods, coming from convex analysis, Hamilton–Jacobi partial differential equations, metric geometry, Perron-Frobenius and nonlinear fixed-point theories, combinatorics or algorithmic complexity. The emphasis of the project is on mathematical modelling and computational aspects.

The subtitle of the *Tropical* project, namely, "structures, algorithms, and interactions", refers to the spirit of our research, including a methodological component, computational aspects, and finally interactions with other scientific fields or real world applications, in particular through mathematical modelling.

2.2. Scientific context

Tropical algebra, geometry, and analysis have enjoyed spectacular development in recent years. Tropical structures initially arose to solve problems in performance evaluation of discrete event systems [56], combinatorial optimization [61], or automata theory [107]. They also arose in mathematical physics and asymptotic analysis [96], [93]. More recently, these structures have appeared in several areas of pure mathematics, in particular in the study of combinatorial aspects of algebraic geometry [82], [121], [110], [87], in algebraic combinatorics [75], and in arithmetics [67]. Also, further applications of tropical methods have appeared, including optimal control [100], program invariant computation [48] and timed systems verification [95], and zero-sum games [2].

The term 'tropical' generally refers to algebraic structures in which the laws originate from optimization processes. The prototypical tropical structure is the max-plus semifield, consisting of the real numbers, equipped with the maximum, thought of as an additive law, and the addition, thought of as a multiplicative law. Tropical objects appear as limits of classical objects along certain deformations ("log-limits sets" of Bergman, "Maslov dequantization", or "Viro deformation"). For this reason, the introduction of tropical tools often yields new insights into old familiar problems, leading either to counterexamples or to new methods and results; see for instance [121], [102]. In some applications, like optimal control, discrete event systems, or static analysis of programs, tropical objects do not appear through a limit procedure, but more directly as a modelling or computation/analysis tool; see for instance [117], [56], [85], [62].

Tropical methods are linked to the fields of positive systems and of metric geometry [104], [12]. Indeed, tropically linear maps are monotone (a.k.a. order-preserving). They are also nonexpansive in certain natural metrics (sup-norm, Hopf oscillation, Hilbert's projective metric, ...). In this way, tropical dynamical systems appear to be special cases of nonexpansive, positive, or monotone dynamical systems, which are studied as part of linear and non-linear Perron-Frobenius theory [94], [3]. Such dynamical systems are of fundamental importance in the study of repeated games [101]. Monotonicity properties are also essential in the understanding of the fixed points problems which determine program invariants by abstract interpretation [69]. The latter problems are actually somehow similar to the ones arising in the study of zero-sum games; see [7]. Moreover, positivity or monotonicity methods are useful in population dynamics, either in a discrete space setting [119] or in a PDE setting [57]. In such cases, solving tropical problems often leads to solutions or combinatorial insights on classical problems involving positivity conditions (e.g., finding equilibria of dynamical systems with nonnegative coordinates, understanding the qualitative and quantitative behavior of growth rates / Floquet eigenvalues [10], etc). Other applications of Perron-Frobenius theory originate from quantum information and control [109], [115].

3. Research Program

3.1. Optimal control and zero-sum games

The dynamic programming approach allows one to analyze one or two-player dynamic decision problems by means of operators, or partial differential equations (Hamilton–Jacobi or Isaacs PDEs), describing the time evolution of the value function, i.e., of the optimal reward of one player, thought of as a function of the initial state and of the horizon. We work especially with problems having long or infinite horizon, modelled by stopping problems, or ergodic problems in which one optimizes a mean payoff per time unit. The determination of optimal strategies reduces to solving nonlinear fixed point equations, which are obtained either directly from discrete models, or after a discretization of a PDE.

The geometry of solutions of optimal control and game problems Basic questions include, especially for stationary or ergodic problems, the understanding of existence and uniqueness conditions for the solutions of dynamic programming equations, for instance in terms of controllability or ergodicity properties, and more generally the understanding of the structure of the full set of solutions of stationary Hamilton–Jacobi PDEs and of the set of optimal strategies. These issues are already challenging in the one-player deterministic case, which is an application of choice of tropical methods, since the Lax-Oleinik semigroup, i.e., the evolution semigroup of the Hamilton-Jacobi PDE, is a linear operator in the tropical sense. Recent progress in the deterministic case has been made by combining dynamical systems and PDE techniques (weak KAM theory [72]), and also using metric geometry ideas (abstract boundaries can be used to represent the sets of solutions [86], [4]). The two player case is challenging, owing to the lack of compactness of the analogue of the Lax-Oleinik semigroup and to a richer geometry. The conditions of solvability of ergodic problems for games (for instance, solvability of ergodic Isaacs PDEs), and the representation of solutions are only understood in special cases, for instance in the finite state space case, through tropical geometry and non-linear Perron-Frobenius methods [38], [41], [3].

Algorithmic aspects: from combinatorial algorithms to the attenuation of the curse of dimensionality Our general goal is to push the limits of solvable models by means of fast algorithms adapted to large scale instances. Such instances arise from discrete problems, in which the state space may so large that it is only accessible through local oracles (for instance, in some web ranking applications, the number of states may be the number of web pages) [73]. They also arise from the discretization of PDEs, in which the number of states grows exponentially with the number of degrees of freedom, according to the “curse of dimensionality”. A first line of research is the development of *new approximation methods for the value function*. So far, classical approximations by linear combinations have been used, as well as approximation by suprema of linear or quadratic forms, which have been introduced in the setting of dual dynamic programming and of the so called “max-plus basis methods” [74]. We believe that more concise or more accurate approximations may be obtained by unifying these methods. Also, some max-plus basis methods have been shown to *attenuate the curse of dimensionality* for very special problems (for instance involving switching) [97], [78]. This suggests that the complexity of control or games problems may be measured by more subtle quantities that the mere number of states, for instance, by some forms of metric entropy (for example, certain large scale problems have a low complexity owing to the presence of decomposition properties, “highway hierarchies”, etc.). A second line of of our research is the development of *combinatorial algorithms*, to solve large scale zero-sum two-player problems with discrete state space. This is related to current open problems in algorithmic game theory. In particular, the existence of polynomial-time algorithms for games with ergodic payment is an open question. See e.g. [43] for a polynomial time average complexity result derived by tropical methods. The two lines of research are related, as the understanding of the geometry of solutions allows to develop better approximation or combinatorial algorithms.

3.2. Non-linear Perron-Frobenius theory, nonexpansive mappings and metric geometry

Several applications (including population dynamics [10] and discrete event systems [56], [64], [46]) lead to studying classes of dynamical systems with remarkable properties: preserving a cone, preserving an order, or being nonexpansive in a metric. These can be studied by techniques of non-linear Perron-Frobenius theory [3] or metric geometry [11]. Basic issues concern the existence and computation of the "escape rate" (which determines the throughput, the growth rate of the population), the characterizations of stationary regimes (non-linear fixed points), or the study of the dynamical properties (convergence to periodic orbits). Nonexpansive mappings also play a key role in the "operator approach" to zero-sum games, since the one-day operators of games are nonexpansive in several metrics, see [8].

3.3. Tropical algebra and convex geometry

The different applications mentioned in the other sections lead us to develop some basic research on tropical algebraic structures and in convex and discrete geometry, looking at objects or problems with a "piecewise-linear " structure. These include the geometry and algorithmics of tropical convex sets [49], [40], tropical semialgebraic sets [52], the study of semi-modules (analogues of vector spaces when the base field is replaced by a semi-field), the study of systems of equations linear in the tropical sense, investigating for instance the analogues of the notions of rank, the analogue of the eigenproblems [42], and more generally of systems of tropical polynomial equations. Our research also builds on, and concern, classical convex and discrete geometry methods.

3.4. Tropical methods applied to optimization, perturbation theory and matrix analysis

Tropical algebraic objects appear as a deformation of classical objects thought various asymptotic procedures. A familiar example is the rule of asymptotic calculus,

$$e^{-a/\epsilon} + e^{-b/\epsilon} \asymp e^{-\min(a,b)/\epsilon} \quad , \qquad e^{-a/\epsilon} \times e^{-b/\epsilon} = e^{-(a+b)/\epsilon} \quad , \tag{1}$$

when $\epsilon \to 0^+$. Deformations of this kind have been studied in different contexts: large deviations, zero-temperature limits, Maslov's "dequantization method" [96], non-archimedean valuations, log-limit sets and Viro's patchworking method [122], etc.

This entails a relation between classical algorithmic problems and tropical algorithmic problems, one may first solve the $\epsilon = 0$ case (non-archimedean problem), which is sometimes easier, and then use the information gotten in this way to solve the $\epsilon = 1$ (archimedean) case.

In particular, tropicalization establishes a connection between polynomial systems and piecewise affine systems that are somehow similar to the ones arising in game problems. It allows one to transfer results from the world of combinatorics to "classical" equations solving. We investigate the consequences of this correspondence on complexity and numerical issues. For instance, combinatorial problems can be solved in a robust way. Hence, situations in which the tropicalization is faithful lead to improved algorithms for classical problems. In particular, scalings for the polynomial eigenproblems based on tropical preprocessings have started to be used in matrix analysis [80], [84].

Moreover, the tropical approach has been recently applied to construct examples of linear programs in which the central path has an unexpectedly high total curvature [44], and it has also led to positive polynomial-time average case results concerning the complexity of mean payoff games. Similarly, we are studying semidefinite programming over non-archimedean fields [52], [51], with the goal to better understand complexity issues in classical semidefinite and semi-algebraic programming.

4. Application Domains

4.1. Discrete event systems (manufacturing systems, networks)

One important class of applications of max-plus algebra comes from discrete event dynamical systems [56]. In particular, modelling timed systems subject to synchronization and concurrency phenomena leads to studying dynamical systems that are non-smooth, but which have remarkable structural properties (nonexpansiveness in certain metrics , monotonicity) or combinatorial properties. Algebraic methods allow one to obtain analytical expressions for performance measures (throughput, waiting time, etc). A recent application, to emergency call centers, can be found in [46].

4.2. Optimal control and games

Optimal control and game theory have numerous well established applications fields: mathematical economy and finance, stock optimization, optimization of networks, decision making, etc. In most of these applications, one needs either to derive analytical or qualitative properties of solutions, or design exact or approximation algorithms adapted to large scale problems.

4.3. Operations Research

We develop, or have developed, several aspects of operations research, including the application of stochastic control to optimal pricing, optimal measurement in networks [112]. Applications of tropical methods arise in particular from discrete optimization [62], [63], scheduling problems with and-or constraints [103], or product mix auctions [120].

4.4. Computing program and dynamical systems invariants

A number of programs and systems verification questions, in which safety considerations are involved, reduce to computing invariant subsets of dynamical systems. This approach appears in various guises in computer science, for instance in static analysis of program by abstract interpretation, along the lines of P. and R. Cousot [69], but also in control (eg, computing safety regions by solving Isaacs PDEs). These invariant sets are often sought in some tractable effective class: ellipsoids, polyhedra, parametric classes of polyhedra with a controlled complexity (the so called “templates” introduced by Sankaranarayanan, Sipma and Manna [113]), shadows of sets represented by linear matrix inequalities, disjunctive constraints represented by tropical polyhedra [48], etc. The computation of invariants boils down to solving large scale fixed point problems. The latter are of the same nature as the ones encountered in the theory of zero-sum games, and so, the techniques developed in the previous research directions (especially methods of monotonicity, nonexpansiveness, discretization of PDEs, etc) apply to the present setting, see e.g. [76], [81] for the application of policy iteration type algorithms, or for the application for fixed point problems over the space of quadratic forms [7]. The problem of computation of invariants is indeed a key issue needing the methods of several fields: convex and nonconvex programming, semidefinite programming and symbolic computation (to handle semialgebraic invariants), nonlinear fixed point theory, approximation theory, tropical methods (to handle disjunctions), and formal proof (to certify numerical invariants or inequalities).

5. Highlights of the Year

5.1. Highlights of the Year

5.1.1. Notable article

The results of the article [45], providing an unexpected counter example to the "continuous analogue of the Hirsch conjecture", showing that log-barrier interior point methods are not strongly polynomial, have been discussed by Jesus De Loera in his survey of recent advances on Linear Programming, "Algebraic and Topological Tools in Linear Optimization", Notices de l'AMS (volume 66, number 7, 2019, especially pp. 1028-1032.

5.1.2. Awards

Maxime Grangereau (PhD student) has been laureate of the programme "Siebel Scholar 2020", https://twitter.com/polytechnique/status/1177111371835695104

6. New Software and Platforms

6.1. Coq-Polyhedra

KEYWORDS: Coq - Polyhedra - Automated theorem proving - Linear optimization

SCIENTIFIC DESCRIPTION: Coq-Polyhedra is a library providing a formalization of convex polyhedra in the Coq proof assistant. While still in active development, it provides an implementation of the simplex method, and already handles the basic properties of polyhedra such as emptiness, boundedness, membership. Several fundamental results in the theory of convex polyhedra, such as Farkas Lemma, duality theorem of linear programming, and Minkowski Theorem, are also formally proved.

The formalization is based on the Mathematical Components library, and makes an extensive use of the boolean reflection methodology.

FUNCTIONAL DESCRIPTION: Coq-Polyhedra is a library which aims at formalizing convex polyhedra in Coq

NEWS OF THE YEAR: Coq-Polyhedra now provides most of the basic operations on polyhedra. They are expressed on a quotient type that avoids reasoning with particular inequality representations. They include : * the construction of elementary polyhedra (half-spaces, hyperplanes, affine spaces, orthants, simplices, etc) * basic operations such as intersection, projection (thanks to the formalization of the Fourier-Motzkin algorithm), image under linear functions, computations of convex hulls, finitely generated cones, etc. * computation of affine hulls of polyhedra, as well as their dimension

Thanks to this, we have made huge progress on the formalization of the combinatorics of polyhedra. The poset of faces, as well as its fundamental properties (lattice, gradedness, atomicity and co-atomicity, etc) are now formalized. The manipulation of the faces is based on an extensive use of canonical structures, that allows to get the most appropriate inequality representations for reasoning. In this way, we arrive at very concise and elegant proofs, closer to the pen-and-paper ones.

- Participants: Xavier Allamigeon, Vasileios Charisopoulos, Ricardo Katz and Pierre-Yves Strub
- Partners: CIFASIS - Ecole Polytechnique
- Contact: Xavier Allamigeon
- Publications: A Formalization of Convex Polyhedra Based on the Simplex Method - A Formalization of Convex Polyhedra Based on the Simplex Method - First steps in the formalization of convex polyhedra in Coq
- URL: https://github.com/nhojem/Coq-Polyhedra

6.2. EmergencyEval

KEYWORDS: Dynamic Analysis - Simulation - Ocaml - Emergency - Firefighters - Police

SCIENTIFIC DESCRIPTION: This software aims at enabling the definition of a Petri network execution semantic, as well as the instanciation and execution of said network using the aforedefined semantic.

The heart of the project dwells in its kernel which operates the step-by-step execution of the network, obeying rules provided by an oracle. This user-defined and separated oracle computes the information necessary to the kernel for building the next state using the current state. The base of our software is the framework for the instanciation and execution of Petri nets, without making assumptions regarding the semantic.

In the context of the study of the dynamics of emergency call centers, a second part of this software is the definition and implementation of the semantic of call centers modelized as Petri nets, and more specifically timed prioritized Petri nets. A module interoperating with the kernel enables to include all the operational specificities of call centers (urgency level, discriminating between operators and callers ...) while guaranteeing the genericity of the kernal which embeds the Petri net formalism as such.

FUNCTIONAL DESCRIPTION: In order to enable the quantitative study of the throughput of calls managed by emergency center calls and the assesment of various organisationnal configurations considered by the stakeholders (firefighters, police, medical emergency service of the 75, 92, 93 and 94 French departments), this software modelizes their behaviours by resorting to extensions of the Petri net formalism. Given a call transfer protocol in a call center, which corresponds to a topology and an execution semantic of a Petri net, the software generates a set of entering calls in accord with the empirically observed statistic ditributions (share of very urgent calls, conversation length), then simulates its management by the operators with respect to the defined protocol. Transitional regimes phenomenons (peak load, support) which are not yet handled by mathematical analysis could therefore be studied. The ouput of the software is a log file which is an execution trace of the simulation featuring extensive information in order to enable the analysis of the data for providing simulation-based insights for decision makers.

The software relies on a Petri net simulation kernel designed to be as modular and adaptable as possible, fit for simulating other Petri-net related phenomenons, even if their semantic differ greatly.

- Participants: Baptiste Colin and Xavier Allamigeon
- Contact: Baptiste Colin

7. New Results

7.1. Optimal control and zero-sum games

7.1.1. Fixed points of order preserving homogeneous maps and zero-sum games

Participants: Marianne Akian, Stéphane Gaubert.

In a series of joint works with Antoine Hochart, applied methods of non-linear fixed point theory to zero-sum games.

A key issue is the solvability of the ergodic equation associated to a zero-sum game with finite state space, i.e., given a dynamic programming operator T associated to an undiscounted problem, one looks for a vector u, called the bias, and for a scalar λ, the ergodic constant, such that $T(u) = \lambda e + u$. The bias vector is of interest as it allows to determine optimal stationnary strategies.

In [41], we studied zero-sum games with perfect information and finite action spaces, and showed that the set of payments for which the bias vector is not unique (up to an additive constant) coincides with the union of lower dimensional cells of a polyhedral complex, in particular, the bias vector is unique, generically. We provided an application to perturbation schemes in policy iteration.

In [14], we apply game theory methods to the study of the nonlinear eigenproblem for homogeneous order preserving self maps of the interior of the cone. We show that the existence and uniqueness of an eigenvector is governed by combinatorial conditions, involving dominions (sets of states “controlled” by one of the two players). In this way, we characterize the situation in which the existence of an eigenvector holds independently of perturbations, and we solve an open problem raised in [77].

7.1.2. Nonlinear fixed point methods to compute joint spectral raddi of nonnegative matrices

Participant: Stéphane Gaubert.

In [21], we introduce a non-linear fixed point method to approximate the joint spectral radius of a finite set of nonnegative matrices. We show in particular that the joint spectral radius is the limit of the eigenvalues of a family of non-linear risk-sensitive type dynamic programming operators. We develop a projective version of Krasnoselskii-Mann iteration to solve these eigenproblems, and report experimental results on large scale instances (several matrices in dimensions of order 1000 within a minute).

7.1.3. Probabilistic and max-plus approximation of Hamilton-Jacobi-Bellman equations

Participant: Marianne Akian.

We consider fully nonlinear Hamilton-Jacobi-Bellman equations associated to diffusion control problems with finite horizon involving a finite set-valued (or switching) control and possibly a continuum-valued control. In [36], we constructed a lower complexity probabilistic numerical algorithm by combining the idempotent expansion properties obtained by McEneaney, Kaise and Han [92], [98] for solving such problems with a numerical probabilistic method such as the one proposed by Fahim, Touzi and Warin [70] for solving some fully nonlinear parabolic partial differential equations, when the volatility does not oscillate too much. In [37] and [27], we improved the method of Fahim, Touzi and Warin by introducing probabilistic schemes which are monotone without any restrictive condition, allowing one to solve fully nonlinear parabolic partial differential equations with general volatilities. We studied the convergence and obtain error estimates when the parameters and the value function are bounded.

7.1.4. Tropical-SDDP algorithms for stochastic control problems involving a switching control

Participants: Marianne Akian, Duy Nghi Benoît Tran.

The PhD thesis of Benoît Tran, supervised by Jean-Philippe Chancelier (ENPC) and Marianne Akian concerns the numerical solution of the dynamic programming equation of discrete time stochastic control problems.

Several methods have been proposed in the litterature to bypass the curse of dimensionality difficulty of such an equation, by assuming a certain structure of the problem. Examples are the max-plus based method of McEneaney [99], [100], the stochastic max-plus scheme proposed by Zheng Qu [108], the stochastic dual dynamic programming (SDDP) algorithm of Pereira and Pinto [105], the mixed integer dynamic approximation scheme of Philpott, Faisal and Bonnans [55], the probabilistic numerical method of Fahim, Touzi and Warin [70]. We propose to associate and compare these methods in order to solve more general structures.

In a first work [35], see also [24], we build a common framework for both the SDDP and a discrete time and finite horizon version of Zheng Qu’s algorithm for deterministic problems involving a finite set-valued (or switching) control and a continuum-valued control. We propose an algorithm that generates monotone approximations of the value function as a pointwise supremum, or infimum, of basic (affine or quadratic for example) functions which are randomly selected. We give sufficient conditions that ensure almost sure convergence of the approximations to the value function. More recently, we study generalizations of these algorithms to the case of stochastic optimal control problems.

In a recent work, we introduce and study an entropic relaxation of the Nested Distance introduced by Pflug [106].

7.1.5. A variance reduction deflated value iteration algorithm to solve ergodic games

Participants: Marianne Akian, Stéphane Gaubert, Omar Saadi.

Recently, Sidford et al. introduced in [116] a variance reduced value iteration algorithm to solve discounted Markov decision processes. In [25], in a joint work work with Zheng Qu (Hong Kong University), we extended this algorithm to the ergodic (mean payoff) case, and also to the two-player case, exploiting techniques from non-linear spectral theory [39] and variational analysis. The deterministic version of this algorithm also yields a new method (alternative to relative value iteration) to solve ergodic problems.

7.2. Non-linear Perron-Frobenius theory, nonexpansive mappings and metric geometry

7.2.1. *Order isomorphisms and antimorphisms on cones*

Participant: Cormac Walsh.

We have been studying non-linear operators on ordered vector spaces that preserve or reverse the order structure. A bijective map that preserves the order in both directions is called an order isomorphism, and one that reverse the order in both directions is called an order antimorphism. These maps are closely related to the isometries of the Hilbert and Thompson metrics on the interior of the cone of positive elements.

The study of the order isomorphisms of an ordered vector space goes back to Alexandrov and Zeeman, who considered maps preserving the light cone that arises in special relativity. This work was extended to more general cones by Rothaus; Noll and Schäffer; and Artstein-Avidan and Slomka. It was shown, in the finite-dimensional case, that all isomorphisms are affine if the cone has no one-dimensional factors. There are also some results in infinite dimension—however these are unsatisfactory because of the strong assumptions that must be made in order to get the finite-dimensional techniques to work. For example, a typical assumption is that the positive cone is the convex hull of its extreme rays, which is overly restrictive in infinite dimension.

In a recent preprint [34], we broaden the scope of these results, requiring only very mild assumptions, namely that the spaces involved are *complete order unit spaces*. These are ordered vector spaces whose cone of positive elements is Archimedean, and that have an order unit, such that the norm induced by this order unit is complete. We show that the existence of an order isomorphism between two such spaces implies that they are in fact linearly isomorphic as ordered vector spaces.

In addition, we introduce a necessary and sufficient criterion for all order isomorphisms on a complete order-unit space to be affine. This criterion is in terms of the geometry of the dual cone. In the current setting, the dual cone has a cross-section called the state space, whose extreme points are called pure states. The closure of the set of pure states is known as the pure state space. The criterion is then that the union of the supports of the affine dependencies supported by the pure state space is dense in the pure state space.

7.2.2. *Generalization of the Hellinger distance*

Participant: Stéphane Gaubert.

In [58] (joint work with Rajendra Bhatia of Ashoka University and Tanvi Jain, Indian Statistic Institute, New Delhi), we study some generalizations of the Hellinger distance to the space of positive definite matrices.

7.2.3. *Spectral inequalities for nonnegative tensors and their tropical analogues*

Participant: Stéphane Gaubert.

In [30] (joint work with Shmuel Friedland, University of Illinois at Chicago) we extend some characterizations and inequalities for the eigenvalues of nonnegative matrices, such as Donsker-Varadhan, Friedland-Karlin, Karlin-Ost inequalities, to nonnegative tensors. These inequalities are related to a correspondence between nonnegative tensors and ergodic control: the logarithm of the spectral radius of a tensor is given by the value of an ergodic problem in which instantaneous payments are given by a relative entropy. Some of these inequalities involve the tropical spectral radius, a limit of the spectral radius which we characterize combinatorially as the value of an ergodic Markov decision process.

7.3. Tropical algebra and convex geometry

7.3.1. Formalizing convex polyhedra in Coq

Participant: Xavier Allamigeon.

This work is joint with Ricardo Katz (Conicet, Argentina) and Pierre-Yves Strub (LIX, Ecole Polytechnique).

In [54], we have made the first steps of a formalization of the theory of convex polyhedra in the proof assistant Coq. The originality of our approach lies in the fact that our formalization is carried out in an effective way, in the sense that the basic predicates over polyhedra (emptiness, boundedness, membership, etc) are defined by means of Coq programs. All these predicates are then proven to correspond to the usual logical statements. The latter take the form of the existence of certificates: for instance, the emptiness of a polyhedron is shown to be equivalent to the existence of a certificate *a la* Farkas. This equivalence between Boolean predicates and formulas living in the kind `Prop` is implemented by using the boolean reflection methodology, and the supporting tools provided by the Mathematical Components library and its tactic language. The benefit of the effective nature of our approach is demonstrated by the fact that we easily arrive at the proof of important results on polyhedra, such as several versions of Farkas Lemma, duality theorem of linear programming, separation from convex hulls, Minkowski Theorem, etc.

Our effective approach is made possible by implementing the simplex method inside Coq, and proving its correctness and termination. Two difficulties need to be overcome to formalize it. On the one hand, we need to deal with its termination. More precisely, the simplex method iterates over the so-called bases. Its termination depends on the specification of a pivoting rule, whose aim is to determine, at each iteration, the next basis. In this work, we have focused on proving that the lexicographic rule ensures termination. On the other hand, the simplex method is actually composed of two parts. The part that we previously described, called Phase II, requires an initial basis to start with. Finding such a basis is the purpose of Phase I. It consists in building an extended problem (having a trivial initial basis), and applying to it Phase II. Both phases need to be formalized to obtain a fully functional algorithm.

The most recent advances on the project are described in the software section.

7.3.2. Tropical totally positive matrices and planar networks

Participant: Stéphane Gaubert.

In [79] (joint work with Adi Niv) we characterized the tropical analogues of totally positive and totally non-negative matrices, i.e, the images by the valuation of the corresponding classes of matrices over a non-archimedean field. We showed in particular that tropical totally positive matrices essentially coincide with the Monge matrices (defined by the positivity of 2×2 tropical minors), arising in optimal transport, and compare the set of tropical totally positive matrices with the tropicalization of the totally positive Grassmannian. A fundamental property of classical totally positive matrices is their representation as weight matrices of planar network; in the recent work [31], we studied the tropical analogue of this property.

7.3.3. Linear algebra over systems

Participants: Marianne Akian, Stéphane Gaubert.

In a joint work with Louis Rowen (Univ. Bar Ilan), we study linear algebra and convexity properties over “systems”. The latter provide a general setting encompassing extensions of the tropical semifields and hyperfields. A first account of this work was presented by Marianne Akian and Louis Rowen at the SIAM conference on applied algebraic geometry, in Bern.

7.3.4. Ambitropical convexity and Shapley retracts

Participants: Marianne Akian, Stéphane Gaubert.

Closed tropical convex cones are the most basic examples of modules over the tropical semifield. They coincide with sub-fixed-point sets of Shapley operators – dynamic programming operators of zero-sum games. We study a larger class of cones, which we call "ambitropical" as it includes both tropical cones and their duals. Ambitropical cones can be defined as lattices in the order induced by Rn. Closed ambitropical cones are precisely the fixedpoint sets of Shapley operators. They are characterized by a property of best co-approximation arising from the theory of nonexpansive retracts of normed spaces. Finitely generated ambitropical cones arise when considering Shapley operators of deterministic games with finite action spaces. Finitely generated ambitropical cones are special polyhedral complexes whose cells are alcoved poyhedra, and locally, they are in bijection with order preserving retracts of the Boolean cube This is a joint work with Sara Vannucci (invited PhD student from Salerno university). A first account of this work was presented by Stéphane Gaubert at the JAMI Workshop, Riemann-Roch theorem in characteristic one and related topics, in Baltimore.

7.3.5. Volume and integer points of tropical polytopes

Participant: Stéphane Gaubert.

We investigate in [20] (joint work with Marie McCaig) the volume of tropical polytopes, as well as the number of integer points contained in integer polytopes. We proved that even approximating these values for a tropical polytope given by its vertices is hard, with no approximation algorithm with factor $2^{\text{poly}(m,n)}$ existing unless $\text{P} = \text{NP}$.

7.4. Tropical methods applied to optimization, perturbation theory and matrix analysis

7.4.1. Tropicalization of semidefinite programming and its relation with stochastic games

Participants: Xavier Allamigeon, Stéphane Gaubert.

Semidefinite programming consists in optimizing a linear function over a spectrahedron. The latter is a subset of $\mathbb{R}^n$ defined by linear matrix inequalities, i.e., a set of the form

$$\left\{x \in \mathbb{R}^n : Q^{(0)} + x_1 Q^{(1)} + \cdots + x_n Q^{(n)} \succeq 0\right\}$$

where the $Q^{(k)}$ are symmetric matrices of order m, and $\succeq$ denotes the Loewner order on the space of symmetric matrices. By definition, $X \succeq Y$ if and only if $X - Y$ is positive semidefinite.

Semidefinite programming is a fundamental tool in convex optimization. It is used to solve various applications from engineering sciences, and also to obtain approximate solutions or bounds for hard problems arising in combinatorial optimization and semialgebraic optimization.

A general issue in computational optimization is to develop combinatorial algorithms for semidefinite programming. Indeed, semidefinite programs are usually solved via interior point methods. However, the latter provide an approximate solution in a polynomial number of iterations, provided that a strictly feasible initial solution. Semidefinite programming becomes a much harder matter if one requires an exact solution. The feasibility problem belongs to $\mathsf{NP}_{\mathbb{R}} \cap \mathsf{coNP}_{\mathbb{R}}$, where the subscript $\mathbb{R}$ refers to the BSS model of computation. It is not known to be in NP in the bit model.

The PhD thesis of Mateusz Skomra [118] dealt about semidefinite programming, in the case where the field $\mathbb{R}$ is replaced by a nonarchimedean field, like the field of Puiseux series. In this case, methods from tropical geometry can be applied and are expected to allow one, in generic situations, to reduce semialgebraic problems to combinatorial problems, involving only the nonarchimedean valuations (leading exponents) of the coefficients of the input.

To this purpose, we studied tropical spectrahedra, which are defined as the images by the valuation of nonarchimedean spectrahedra. We establish that they are closed semilinear sets, and that, under a genericity condition, they are described by explicit inequalities expressing the nonnegativity of tropical minors of order 1 and 2. These results are presented in the preprint [52] (now accepted for publication in Disc. Comp. Geom), with further results in the PhD thesis [118].

We showed in [53] that the feasibility problem for a generic tropical spectrahedron is equivalent to solving a stochastic mean payoff game (with perfect information). The complexity of these games is a long-standing open problem. They are not known to be polynomial, however they belong to the class $\mathsf{NP} \cap \mathsf{coNP}$, and they can be solved efficiently in practice. This allows to apply stochastic game algorithms to solve nonarchimedean semidefinite feasibility problems. We obtain in this way both theoretical bounds and a practicable method which solves some large scale instances.

A long-standing problem is to characterize the convex semialgebraic sets that are SDP representable, meaning that they can be represented as the image of a spectrahedron by a (linear) projector. Helton and Nie conjectured that every convex semialgebraic set over the field of real numbers are SDP representable. Recently, [114] disproved this conjecture. In [15], we show, however, that the following result, which may be thought of as a tropical analogue of this conjecture, is true: over a real closed nonarchimedean field of Puiseux series, the convex semialgebraic sets and the projections of spectrahedra have precisely the same images by the nonarchimedean valuation. The proof relies on game theory methods and on our previous results [52] and [53].

In [50] and [118], we exploited the tropical geometry approach to introduce a condition number for stochastic mean payoff games (with perfect information). This condition number is defined as the maximal radius of a ball in Hilbert's projective metric, contained in a primal or dual feasible set. We show that the convergence time of value iteration is governed by this condition number, and derive fixed parameter tractability results.

7.4.2. Tropical polynomial systems and colorful interior of convex bodies

Participants: Marianne Akian, Marin Boyet, Xavier Allamigeon, Stéphane Gaubert.

We studied tropical polynomial systems, with motivations from call center performance evaluation (see Section 7.6.1). We introduced a notion of colorful interior of a family of convex bodies, and showed that the solution of such a polynomial system reduces to linear programming if one knows a vector in the colorful interior of an associated family of Newton polytopes. Further properties of colorful interiors were investigated, as well as the relation between tropical colorful interiors and support vector machines. These results were presented by M. Boyet at the SIAM AG conference in Bern.

7.4.3. Universal approximation theorems by log-sum-exp neural networks

Participant: Stéphane Gaubert.

This is a joint work with Giuseppe Calafiore and Corrado Possieri (Torino).

We establish universal properties of functions by neural networks with log-sum-exp activation functions, first for convex functions [19], and then in general [29]. Some consequences, including approximation by subtraction free rational expressions, are derived.

7.5. Tropical algebra, number theory and directed algebraic topology

7.5.1. An arithmetic site of Connes-Consani type for number fields with narrow class number 1

Participant: Aurélien Sagnier.

In 1995, A. Connes ([65]) gave a spectral interpretation of the zeroes of the Riemann zeta function involving the action of $\mathbb{R}_+^*$ on the sector $X = \mathbb{Q}_+^\times \backslash \mathbb{A}_\mathbb{Q}/\widehat{\mathbb{Z}}^\times$ of the adele class space $\mathbb{A}_\mathbb{Q}/\mathbb{Q}^*$ of the field of rational numbers. In [66], [68], the action of $\mathbb{R}_+^*$ on this sector X was shown to have a natural interpretation in algebraic geometry. This interpretation requires the use of topos theory as well as of the key ingredient of characteristic one namely the semifield $\mathbb{R}_{\max}$ familiar in tropical geometry. The automorphism group of this semifield is naturally isomorphic to $\mathbb{R}_+^*$ and plays the role of the Frobenius. As it turns out, its action on the points of a natural semiringed topos corresponds canonically to the above action on X. This semiringed topos is called the arithmetic site. In my PhD, I extended the construction of the arithmetic site, replacing the field of rational numbers by certain number fields. I considered the simplest complex case, namely that of imaginary quadratic fields on which we assume that the units are not reduced to ± 1 that is when K is either $\mathbb{Q}(\imath)$ or $\mathbb{Q}(\imath\sqrt{3})$. In particular, during this year, we showed that the semiring of convex polygons introduced for those cases satisfies a sublte arithmetical universal property. These results are presented in the accepted in *Journal of Number Theory* article [111]. In a further work, developed this year, I extended this construction, dealing now with number fields K with narrow class number 1, this generalization will rely on the universal property discovered this year and on the extensive use of Shintani's unit theorem. Here again tropical algebra play a crucial role in the geometrical constructions.

7.5.2. Duality between tropical modules and congruences

Participants: Stéphane Gaubert, Aurélien Sagnier.

In a joint work with Éric Goubault (LIX, École polytechnique), we establish a duality theorem between congruences and modules over tropical semifields.

7.5.3. Directed topological complexity and control

Participant: Aurélien Sagnier.

This is a joint work with Michael Farber and Eric Goubault.

The view we are taking here is that of topological complexity, as defined in [71], adapted to directed topological spaces.

Let us briefly motivate the interest of a directed topological complexity notion. It has been observed that the very important planification problem in robotics boils down to, mathematically speaking, finding a section to the path space fibration $\chi \ : \ PX = X^I \to X \times X$ with $\chi(p) = (p(0), p(1))$. If this section is continuous, then the complexity is the lowest possible (equal to one), otherwise, the minimal number of discontinuities that would encode such a section would be what is called the topological complexity of X. This topological complexity is both understandable algorithmically, and topologically, e.g. as s having a continuous section is equivalent to X being contractible. More generally speaking, the topological complexity is defined as the Schwartz genus of the path space fibration, i.e. is the minimal cardinal of partitions of $X \times X$ into "nice" subspaces F_i such that $s_{F_i} \ : \ F_i \to PX$ is continuous.

This definition perfectly fits the planification problem in robotics where there are no constraints on the actual control that can be applied to the physical apparatus that is supposed to be moved from point a to point b. In many applications, a physical apparatus may have dynamics that can be described as an ordinary differential equation in the state variables $x \in \mathbb{R}^n$ and in time t, parameterized by control parameters $u \in \mathbb{R}^p$, $\dot{x}(t) = f(t, x(t))$. These parameters are generally bounded within some set U, and, not knowing the precise control law (i.e. parameters u as a function of time t) to be applied, the way the controlled system can evolve is as one of the solutions of the differential inclusion $\dot{x}(t) \in F(t, x(t))$ where $F(t, x(t))$ is the set of all $f(t, x(t), u)$ with $u \in U$. Under some classical conditions, this differential inclusion can be proven to have solutions on at least a small interval of time, but we will not discuss this further here. Under the same conditions, the set of solutions of this differential inclusion naturally generates a dspace (a very general structure of directed space, where a preferred subset of paths is singled out, called directed paths, see e.g. [83]). Now, the planification problem in the presence of control constraints equates to finding sections to the analogues to the path space fibration (That would most probably not qualify for being called a fibration in the directed setting) taking a dipath to its end points. This notion is developed in this article, and we introduce a notion of directed homotopy equivalence that has precisely, and in a certain non technical sense, minimally, the right properties with respect to this directed version of topological complexity.

This notion of directed topological complexity also has applications in informatics where a directed space can be used to model the space of all possible executions of a concurrent process (ie when several running programs must share common limited ressources).

In the article [22], after defining the notion of directed topological complexity,this invariant (directed topological complexity) is studied for directed spheres and directed graphs.

7.6. Applications

7.6.1. Performance evaluation of emergency call centers

Participants: Xavier Allamigeon, Marin Boyet, Baptiste Colin, Stéphane Gaubert.

Since 2014, we have been collaborating with Préfecture de Police (Régis Reboul and LcL Stéphane Raclot), more specifically with Brigade de Sapeurs de Pompiers de Paris (BSPP) and Direction de Sécurité de Proximité de l'agglomération parisienne (DSPAP), on the performance evaluation of the new organization (PFAU, "Plate forme d'appels d'urgence") to handle emergency calls to firemen and policemen in the Paris area. We developed analytical models, based on Petri nets with priorities, and fluid limits, see [46], [47], [59]. In 2019, with four students of École polytechnique, Céline Moucer, Julia Escribe, Skandère Sahli and Alban Zammit, we performed case studies, showing the improvement brought by the two level filtering procedure.

Moreover, in 2019, this work was extended to encompass the handling of health emergency calls, with a new collaboration, involving responsibles from the four services of medical emergency aid of Assistance Publique – Hôpitaux de Paris (APHP), i.e., with SAMU75, 92, 93, 94, in the framework of a project led by Dr. Christophe Leroy from APHP. As part of his PhD work, Marin Boyet developed Petri net models capturing the characteristic of the centers (CRRA) handling emergency calls the SAMU, in order to make dimensioning recommendations.

7.6.2. Game theory and optimization methods for decentralized electric systems

Participants: Stéphane Gaubert, Paulin Jacquot.

This work is in collaboration with Nadia Oudjane, Olivier Beaude and Cheng Wan (EDF Lab).

The PhD work of Paulin Jacquot concerns the application of game theory and distributed optimization techniques to the operation of decentralized electric systems, and in particlar to the management of distributed electric consumption flexibilities. We start by adopting the point of view of a centralized operator in charge of the management of flexibilities for several agents. We provide a distributed and privacy-preserving algorithm to compute consumption profiles for agents that are optimal for the operator. In the proposed method, the individual constraints as well as the individual consumption profile of each agent are never revealed to the operator or the other agents [33] [28]. A patent related to this method has been submitted [89].

Then, in a second model, we adopt a more decentralized vision and consider a game theoretic framework for the management of consumption flexibilities. This approach enables, in particular, to take into account the strategic behavior of consumers. Individual objectives are determined by dynamic billing mechanisms, which is motivated by the modeling of congestion effects occurring on time periods receiving a high electricity load from consumers. A relevant class of games in this framework is given by atomic splittable congestion games. We obtain several theoretical results on Nash equilibria for this class of games, and we quantify the efficiency of those equilibria by providing bounds on the price of anarchy. We address the question of the decentralized computation of equilibria in this context by studying the conditions and rates of convergence of the best response and projected gradients algorithms [91], [88].

A fruitful collaboration with Cheng Wan (EDF Lab) led to the third part of this PhD thesis. In this part, we consider an operator dealing with a very large number of players, for which evaluating the equilibria in a congestion game will be difficult. To address this issue, we give approximation results on the equilibria in congestion and aggregative games with a very large number of players, in the presence of coupling constraints. These results, obtained in the framework of variational inequalities and under some monotonicity conditions, can be used to compute an approximate equilibrium, solution of a small dimension problem [32]. In line with the idea of modeling large populations, we consider nonatomic congestion games with coupling constraints, with an infinity of heterogeneous players: these games arise when the characteristics of a population are described by a parametric density function. Under monotonicity hypotheses, we prove that Wardrop equilibria of such games, given as solutions of an infinite dimensional variational inequality, can be approximated by symmetric Wardrop equilibria of auxiliary games, solutions of low dimension variational inequalities. Again, those results can be the basis of tractable methods to compute an approximate Wardrop equilibrium in a nonatomic infinite-type congestion game [33]. Last, in a collaboration with Hélène Le Cadre, Cheng Wan and Clémence Alasseur, we consider a game model for the study of decentralized peer-to-peer energy exchanges between a community of consumers with renewable production sources. We study the generalized equilibria in this game, which characterize the possible energy trades and associated individual consumptions. We compare the equilibria with the centralized solution minimizing the social cost, and evaluate the efficiency of equilibria through the price of anarchy [23].

Paulin Jacquot defended his PhD on December 5, 2019 at Ecole polytechnique [90].

8. Bilateral Contracts and Grants with Industry

8.1. Bilateral Contracts with Industry

- Decentralized mechanisms of operation of power systems: equilibria and efficiency. Collaboration with Nadia Oudjane and Olivier Beaude from EDF-labs, with the PhD work of Paulin Jacquot (CIFRE PhD), supervised by Stéphane Gaubert.
- Stochastic optimization of multiple flexibilities and energies in micro-grids, collaboration with Wim Van Ackooij, from EDF labs, with the PhD work of Maxime Grangereau (CIFRE PhD), supervised by Emmanuel Gobet (CMAP) and cosupervised by Stéphane Gaubert.

9. Partnerships and Cooperations

9.1. National Initiatives

9.1.1. ANR

- Projet ANR JCJC CAPPS (“Combinatorial Analysis of Polytopes and Polyhedral Subdivisions”), responsable Arnau Padrol (IMJ-PRG, Sorbonne Université). Partenaires : IMJ-PRG (Sorbonne Université), Inria Saclay (Tropical), LIGM (Université Paris-Est Marne-la-Vallée), LIF (Université Aix-Marseille), CERMICS (École Nationale des Ponts et Chaussées), LIX (École Polytechnique).

9.1.2. Labex Hadamard

- Projet du Labex Hadamard, intitulé “ALgebraic Methods in gAmes and optimization ALMA”, conjoint avec le PGMO, coordonné par E. Tisgaridas (Inria Paris) et X. Allamigeon, faisant intervenir M. Akian et S. Gaubert.

9.1.3. IRS iCODE (Institut pour le Contrôle et la Décision de l’Idex Paris-Saclay)

- White project “New perspectives in the numerical solution of Hamilton-Jacobi-Bellman partial differential equations”, coordinated by M. Akian, including S. Gaubert and members of the EPC Commands (Inria Saclay and École polytechnique), UMA (ENSTA), and LMO (Paris-Sud).

9.1.4. Centre des Hautes Études du Ministère de l'Intérieur

- Project "Optimisation de la performance de centres de traitement d'appels d'urgence en cas d'événements planifiés ou imprévus", coordinated by X. Allamigeon, involving M. Boyet, B. Colin and S. Gaubert.

9.2. International Initiatives

9.2.1. Participation in Other International Programs

- Bilateral projects FACCTS, between the University of Chicago (Statistics) – Lek-Heng Lim– and Ecole polytechnique – Stéphane Gaubert– "Tropical geometry of deep learning".
- Math AmSud Project ARGO, "Algebraic Real Geometry and Optimization", accepted, with CMM (Chile), Univ. Buenos Aires (Argentina), Univ. Fed. Rio and Univ. Fed. Ceara (Brasil), Univ Savoie and CMAP, Ecole polytechnique (France).

9.3. International Research Visitors

9.3.1. Visits of International Scientists

- Oliver Lorscheid, IMPA, Rio (on sabattical at MPI, Bonn), one week in June and 3 days in October, joint invitation with CMLS, Ecole polytechnique.
- Louis Rowen, Bar Ilan University, 3 days in March.
- Sergei Sergeev, Birmingham, 1 week in April.
- Grigorio Malajovich, Univ. Federal, Rio, 1 week in August.
- Armando Gutiérrez, Aalto University, 2 days in February.

9.3.1.1. Internships

- Sarah Vannucci, PhD student, University of Salerno, has been invited for 3 months in the team.

9.3.2. Visits to International Teams

- S. Gaubert
  - Univ. Birmingham, Math and Stats Dep, Jan. 2019 (visiting S. Sergeev)
  - Univ. Bar Ilan, Math Dep, June 2019 (visiting L. Rowen)
  - Univ. Baltimore, Math Dep, Oct. 2019 (visiting A. Sagnier)
- B. Tran
  - U. de Hong Kong, March-April 2019 (2 months, visiting Zheng Qu)
- C. Walsh
  - Univ. Kent, School of Mathematics, Statistics and Actuarial Science, 1 week in November (visiting B. Lemmens and M. Roelands).

10. Dissemination

10.1. Promoting Scientific Activities

10.1.1. Scientific Events: Organisation

10.1.1.1. General Chair, Scientific Chair

- M. Akian is co-chair of the organizing committee of the next 2020 SMAI MODE days, see http://smai-mode2020.inria.fr/.

- Stéphane Gaubert is the coordinator of the Gaspard Monge Program for Optimization, Operations Research and their interactions with data sciences (PGMO), a corporate sponsorhip program, operated by Fondation Mathématique Jacques Hadamard, supported by Criteo, EDF, Orange and Thales, see http://www.fondation-hadamard.fr/fr/pgmo/.

10.1.1.2. Member of the Organizing Committees

- S. Gaubert co-organizes the "Séminaire Parisien d'Optimisation" at Institut Henri Poincaré. https://sites.google.com/site/spoihp/.
- A. Sagnier co-organized the JAMI Workshop, Riemann-Roch theorem in characteristic one and related topics, Univ. Baltimore, Oct. 2019.

10.1.2. Scientific Events: Selection

10.1.2.1. Chair of Conference Program Committees

- S. Gaubert, Chair of the PGMO days, EDF Lab Paris Saclay, Dec 3-4, 2019. http://www.fondation-hadamard.fr/fr/pgmo/pgmodays

10.1.2.2. Member of the Conference Program Committees

- Marianne Akian is member of the scientific committee of the next 2020 SMAI MODE days, see http://smai-mode2020.inria.fr/.
- S. Gaubert has been a member of the scientific committee of the JAMI Workshop, Riemann-Roch theorem in characteristic one and related topics, Univ. Baltimore, Oct. 2019.

10.1.3. Journal

10.1.3.1. Member of the Editorial Boards

- S. Gaubert is member of the editorial committee of the collection Mathématiques et Applications, SMAI and Springer.
- S. Gaubert is associate editor of Linear and Multilinear Algebra.
- S. Gaubert is associate editor of RAIRO Operations research.
- S. Gaubert is associate editor of the Journal of dynamics and games.

10.1.4. Invited Talks

- M. Akian
  - "Complexity of policy and value iterations for ergodic zero-sum two player games: non-linear Perron-Frobenius methods", Journées annuelles du GDR MOA, Rennes, October 2019.
- S. Gaubert
  - "Condition numbers in nonarchimedean semidefinite programming ...and what they say about stochastic mean payoff games", Tropical Mathematics group of the London Math. Soc, Birmingham, January 2019.
  - "Nonarchimedean convex programming and its relation to mean payoff games", Game theory conference, CIRM, Luminy, June 2019.
- X. Allamigeon
  - "Tropical geometry meets optimization", conference "Network Games, Tropical Geometry, and Quantum Communication", ZIB, Berlin, June 2019

10.1.5. Leadership within the Scientific Community

- See Section 10.1.1.1 (coordination of PGMO).

10.1.6. Research Administration

- M. Akian :

  - Member of the "comité de liaison SMAI-MODE" since June 2015.
  - Member of the laboratory council of CMAP.
  - Member of the scientific board of Inria.
- S. Gaubert :
  - Chairman of the Gaspard Monge Program for Optimization, Operations Research and their interactions with data sciences (PGMO), see 10.1.1.1 for details.
  - Member of the scientific council of CMAP.
- X. Allamigeon:
  - Member of the scientific committee of Inria Saclay – Ile-de-France.
  - Member of the Applied Mathematics Department committee at Ecole Polytechnique.

10.2. Teaching - Supervision - Juries

10.2.1. Teaching

- M. Akian
  - Course "Markov decision processes: dynamic programming and applications" joint between (3rd year of) ENSTA and M2 "Mathématiques et Applications", U. Paris Saclay, "Optimization", and shared with Jean-Philippe Chancelier (ENPC), 15 hours each.
- X. Allamigeon
  - Petites classes et encadrement d'enseignements d'approfondissement de Recherche Opérationnelle en troisième année à l'École Polytechnique (programme d'approfondissement de Mathématiques Appliquées) (niveau M1).
  - Cours du M2 "Optimisation" de l'Université Paris Saclay, cours partagé avec Manuel Ruiz (RTE) et Dominique Quadri (LRI, Université Paris Sud).
  - Co-responsabilité du programme d'approfondissement en mathématiques appliquées (troisième année) à l'École Polytechnique.
- S. Gaubert
  - Course "Systèmes à Événements Discrets", option MAREVA, ENSMP.
  - Course "Algèbre tropicale pour le contrôle optimal et les jeux" of "Contrôle, Optimisation et Calcul des Variations" (COCV) of M2 "Mathématiques et Applications" of Paris 6 University and École Polytechnique.
  - Lecture of Operations Research, third year of École Polytechnique. The lectures notes were published as a book [60].
- O. Saadi
  - Exercises classes in the framework of a "Monitorat".
- B. Tran
  - Exercises classes for the first year of Bachelor program of Ecole polytechnique in the framework of a "Monitorat".

10.2.2. Supervision

- PhD: Paulin Jacquot, registered at Univ. Paris Saclay since November 2016, thesis supervisor: Stéphane Gaubert, cosupervision: Nadia Oujdane, Olivier Beaude (EDF), the defense took place in Dec. 2019.
- PhD in progress: Benoît Tran, registered at Univ Paris-Est Marne La Vallée, since September 2017, thesis supervisor: Jean-Philippe Chancelier (ENPC), cosupervision: Marianne Akian.

- PhD in progress: Maxime Grangereau, registered at Univ. Paris Saclay since Jan 2018, thesis supervisor: Emanuel Gobet, cosupervision: Stéphane Gaubert.
- PhD in progress: Omar Saadi, registered at Univ. Paris Saclay since October 2018, thesis supervisor: Stéphane Gaubert, cosupervision: Marianne Akian.
- PhD in progress: Marin Boyet, registered at Univ. Paris Saclay since October 2018, thesis supervisor: Stéphane Gaubert, cosupervision: Xavier Allamigeon.
- PhD in progress: Maël Forcier, registered at ENPC since September 2019, thesis supervisor: Vincent Leclère, cosupervision Stéphane Gaubert.

10.2.3. Juries

- M. Akian
  - Jury of the 2019 competition for a full professor position in Applied Mathematics at Avignon University.
- S. Gaubert
  - Jury of the 2019 competition for CR positions of Inria Saclay–Île-de-France.
  - Jury of the HdR of Pascale Bendotti (Sorbonne Universités, reviewer, June 2019).
  - Jury of the HdR of Arnau Padrol (Sorbonne Universités, examiner, December 2019).
  - Jury of the PhD thesis of Aiwen Li (Angers, reviewer, Sep 2019).
  - Jury of the PhD thesis of J. Trunk (TU-Berlin, reviewer, Oct 2019),
  - Jury of the PhD thesis of Paul Beaujean (Dauphine, reviewer, Dec. 2019).
  - Jury of the PhD thesis of Paulin Jacquot (Ecole Polytechnique, examiner, Dec. 2019).
  - Jury of the PhD thesis of Arnaud Le Rhun (Ecole Polytechnique, president, Dec. 2019).
- C. Walsh
  - Jury of the PhD thesis of Armando Gutiérrez (Aalto University, pre-examiner, Dec. 2019).

10.3. Conferences, Seminars

- M. Akian
  - “Linear algebra and convexity over symmetrized semirings, hyperfields and systems”, SIAM Conference on Applied Algebraic Geometry, Bern, July 2019.
  - “The operator approach to entropy games”, International Workshop on Operator Theory and Applications (IWOTA), Lisbonne, July 2019.
  - “Linear algebra and convexity over symmetrized semirings, hyperfields and systems”, International Conference on Matrix Analysis and its Applications, MAT TRIAD 2019, Liblice, Czech Republic, September 2019.
  - “A game theory approach to the existence and uniqueness of nonlinear Perron-Frobenius eigenvectors”, French-German-Swiss (FGS) conference in Optimization, September 2019, Nice.
  - “De l’ergodicité des jeux à somme nulle à l’existence et l’unicité de vecteurs propres de Perron non linéaires”, Workshop "Jeux dynamiques: temps discret, temps continu", Fréjus, October 2019.
  - “Complexity of policy and value iterations for ergodic zero-sum two player games: non-linear Perron-Frobenius methods”, Journées annuelles du GDR MOA, Rennes, October 2019.
- X. Allamigeon

  - Condition numbers of stochastic mean payoff games and what they say about nonarchimedean semidefinite programming, SIAM Conference on Applied Algebraic Geometry, Bern, July 2019.
- M. Boyet
  - The shadow vertex algorithm solves colorful one-versus-all tropical polynomial systems, PGMO Days, Nov. 21, 2018, Palaiseau.
- S. Gaubert
  - Condition numbers in nonarchimedean semidefinite programming ...and what they say about stochastic mean payoff games, Tropical Mathematics group of the London Math. Soc, Birmingham, January 2019.
  - Nonarchimedean convex programming and its relation to mean payoff games, Séminaire, Université de Perpignan, March 2019.
  - Nonarchimedean convex programming and its relation to mean payoff games, Game theory conference, CIRM, Luminy, June 2019.
  - Dynamics of priority: from emergency call centers to tropical polynomial systems, SIAM Conference on Applied Algebraic Geometry, Bern, July 2019.
  - Dynamic programming operators over noncommutative spaces: an approach to optimal control of switched systems, International Workshop on Operator Theory and Applications (IWOTA), Lisbonne, July 2019.
  - The operator approach to entropy games, French-German-Swiss conference in Optimization, September 2019, Nice.
  - "Convexité ambitropicale – ou comment caractériser les rétracts de Shapley", Workshop "Jeux dynamiques: temps discret, temps continu", Fréjus, October 2019.
  - From tropical to ambitropical convexity, JAMI Workshop, Riemann Roch in characteristic one and related topics, Baltimore, October 2019.
- P. Jacquot
  - A Privacy-preserving Disaggregation Algorithm for Non-intrusive Management of Flexible Energy, IEEE 58th Conference on Decision and Control, Nice, France, December 2019.
  - Peer-to-Peer Electricity Market Analysis: From Variational to Generalized Nash Equilibrium, PGMO Days, December 2019, EDF Saclay, France.
  - A Privacy-Preserving Disaggregation Algorithm for Nonconvex Optimization based on Alternate Projections, French-German-Swiss Conference on Optimization (FGS), Nice, France, September 2019.
  - Nonatomic Aggregative Games with Infinitely Many Types, Game Theory Seminar, Insitut Henri Poincaré, Paris, France, October 2019.
  - A Privacy-preserving Method to Optimize Distributed Resource Allocation, Journée de rentrée du CMAP (Invited), Ecole polytechnique, France, October 2019.
- A. Sagnier
  - Talk at University of Strasbourg on «An arithmetic site at the complex place», April 2019
  - Talk at University of Antwerp on «An arithmetic site at the complex place»,

    April 2019
  - Short communication at JAMI 2019 «Riemann-Roch in characteristic one and related topics» on «An arithmetic site at the complex place», Johns Hopkins University, Baltimore, October 2019
- B. Tran

  - “A Min-Plus / SDDP Algorithm for Multistage Stochastic Convex Programming”, The XV international conference on stochastic programming (ICSP 2019), Trondheim, July 2019.
  - “A Min-plus-SDDP Algorithm for Multistage Stochastic Convex Programming”, French-German-Swiss Conference on Optimization (FGS), Nice, France, September 2019.
  - “A Min-plus-SDDP Algorithm for Deterministic Multistage Convex Programming”, 58th IEEE Conference on Decision and Control (CDC 2019), Nice, December 2019.
- C. Walsh
  - “For which ordered vector spaces are all order isomorphisms affine linear?”, Advances in the Geometric and Analytic Theory of Convex Cones, Jeju, Korea, May 27–31, 2019.
  - “Approximability of convex bodies and volume growth in Hilbert geometries”, Perspectives on convex projective geometry, Sète, France, 24–28 June 2019.
  - “For which ordered vector spaces are all order isomorphisms affine linear?”, Positivity X, Pretoria, South Africa, 8–12 July, 2019.
  - “For which ordered vector spaces are all order isomorphisms affine linear?”, 30th International Workshop on Operator Theory and its Applications, IWOTA 2019, Lisbon, Portugal, 22–26 July, 2019.

10.4. Popularization

10.4.1. Articles and contents

- The collaboration developed by our team since 2014, with Préfecture de Police, on the performance evaluation of the new organization (PFAU, Plate forme d’appels d’urgences) to handle the calls to the emergency numbers 17-18-112 in the Paris area is described in the following article, published on the Inria web site, https://www.inria.fr/centre/saclay/actualites/gestion-des-appels-d-urgence-une-geometrie-tropicale-pour-les-secours-parisiens

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Major publications by the team in recent years

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Articles in International Peer-Reviewed Journals

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